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Effect of high-frequency magnetic fluctuations on MHD waves

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We have shown earlier that compressional magnetic fluctuations modify the properties of MHD waves. The modifications can be large even when the fluctuation level is much smaller than the ambient magnetic field. There are two kinds of effects: resonant and non-resonant. The former is a result of the so-called parametric cyclotron resonance, and enables energy transfer from the wave to particle Larmor motion. This transfer causes the damping of waves even when their frequency is so small that ordinary cyclotron damping is negligible. The non-resonant effect does not cause energy exchange, but it does modify wave dispersion and causes mode coupling between the shear Alfvén mode and magnetosonic modes.

1. Introduction

The MHD approximation is valid for waves whose temporal and spatial scales are much greater than those of ion Larmor motion. Therefore the MHD approximation is extensively used to study large-scale phenomena in plasmas, such as large-scale energy transfer in the solar wind or three-dimensional modelling of the magnetosphere. Large-scale plasma systems often contain higher-frequency fluctuations, which modify the properties of the large-scale phenomena. For example, high-frequency turbulent waves can give rise to anomalous transport, which acts like resistivity or viscosity on the MHD scale.

In this paper we shall present a new high-frequency fluctuation effect. We shall see that high-frequency (of the order of the ion Larmor frequency) compressional magnetic fluctuations can cause anomalous drifts of ions and consequently modify the MHD wave propagation. The modification is considerable even when the fluctuation amplitude is much smaller than the ambient magnetic field.

When the magnetic field fluctuates in magnitude, the beat frequency between the electric field and the magnetic field fluctuation can resonate with cyclotron motion. This so-called parametric cyclotron resonance (PCR) provides a means to exchange energy between particles and low-frequency waves, and it enables high-frequency magnetic fluctuations to modify the MHD wave propagation (Nakamura & Kennel 1993; Nakamura *et al.* 1993). Nakamura & Kennel (1993) examined the basic PCR mechanism, and described the energy exchange between a shear Alfvén wave and ion Larmor motion as a simple example of its application. Here we study in detail the properties of MHD waves in a background magnetic field that fluctuates in magnitude, and we calculate the dispersion relation of the waves in a more self-consistent manner than in our first paper. Besides the PCR effect, we find that out-of-resonance magnetic fluctuations can have significant effects on MHD waves.

In Nakamura & Kennel (1993) the magnetic field fluctuation was provided by a compressional wave with a simple harmonic structure. However, it is more important for applications to treat magnetic fluctuations with a continuous spectrum, such as turbulent fields caused by nonlinear interaction of many wave modes. In the present paper we study the interaction of three kinds of oscillations; compressional magnetic fluctuation with continuous spectrum, ion Larmor motion and a simple harmonic MHD wave. To obtain the dispersion relation of the MHD wave in the presence of compressional magnetic fluctuations, we make use of a stochastic equation of motion for ions; the magnetic field fluctuations are treated as a stochastic process with given autocorrelation function. Using this stochastic equation of motion, we calculate the ion response to a perpendicular electric field in a fluctuating magnetic field in §2. We obtain a general wave dispersion relation for a cold two-fluid plasma in §3. When this dispersion relation is applied to MHD waves, we find two magnetic fluctuation effects: damping, and coupling between the shear Alfvén mode and the fast magnetosonic mode. These results are presented in §4. We take electron-temperature effects into account in §5 and obtain the dispersion relation in a finite-temperature two-fluid plasma. We discuss the meaning and consistency of our assumption of a background field that fluctuates in magnitude in §6. A brief summary of our results is given in §7.

2. Ion motion in a fluctuating magnetic field

Here we consider two kinds of waves, namely compressional waves with continuous spectrum and a test wave with a simple harmonic structure. We assume the existence of the compressional waves and examine their effects on the propagation of the test wave. In the case of a monochromatic magnetic field fluctuation the drift velocity increases proportionally to time t as long as the wave is in strict resonance. If the wave frequency is slightly out of resonance, the drift velocity becomes a periodic function with a period of $1/\Delta\omega$ (where $\Delta\omega$ is the frequency mismatch); we can none the less say that the wave is 'in resonance' during $t < 1/\Delta\omega$. A continuous spectrum can be considered as the superposition of harmonic waves, and, roughly speaking, wave components with frequency difference $\Delta\omega \lesssim 1/t$ are in resonance at a given time t. Therefore the number of resonant waves decreases inversely proportionally to t, and consequently the drift speed does not increase as fast as in the monochromatic wave resonance. As we shall see in the following, the anomalous drift amplitude becomes constant.

Let us take the z axis parallel to the ambient magnetic field and express a vector perpendicular to the ambient field by a complex number (e.g. $v = v_x + iv_y$). We neglect the spatial variation of the electromagnetic fields, by assuming either sufficiently long wavelength or small displacement of the particle. For the particle's perpendicular motion, the perpendicular magnetic fields of the waves (B_x and B_y) are not important as long as the amplitude is much smaller than the ambient field (Nakamura & Kennel 1993). Therefore we include only the compressional component of the magnetic field. We treat the electric and magnetic field fluctuations of the compressional waves (with $\hat{E}_f(t)$ the perpendicular electric field and $\hat{B}_f(t)$ the parallel magnetic field variation) as stochastic variables; hereinafter we designate stochastic variables by a hat ($\hat{}$).

We suppose a simple harmonic test wave is propagating in this fluctuating plasma with perpendicular electric field $E_T(t)$ and parallel magnetic field variation $B_T(t)$. Then the stochastic equation for ion motion is

$$\begin{aligned}\frac{d\hat{v}}{dt} &= \frac{e}{m_i}\{E_{\mathrm{T}}(t)+\hat{E}_f(t)-i[B_0+\hat{B}_f(t)+B_{\mathrm{T}}]\,\hat{v}\}\\ &\approx \frac{e}{m_i}[E_{\mathrm{T}}(t)+\hat{E}_f(t)]-i[\Omega_i+\hat{\Omega}_f(t)+\Omega_{\mathrm{T}}(t)]\,\hat{v},\end{aligned} \tag{1}$$

$$\Omega_i \equiv \frac{eB_0}{m_i}, \quad \hat{\Omega}_f \equiv \frac{e\hat{B}_f}{m_i}, \quad \Omega_{\mathrm{T}} \equiv \frac{eB_{\mathrm{T}}}{m_i},$$

where m_i and e are the ion mass and charge. Strictly speaking, the electromagnetic fields of the test wave are also stochastic variables with fluctuation because they must be self-consistently calculated from particle orbits that include stochastic effects. However, the fluctuation is very small because the macroscopic wave fields result from the collective motion of plasma particles; the fluctuations of each particle are smeared out in the summation to get macroscopic quantities. Therefore we can treat the test wave field as an ordinary rather than stochastic function. Equation (1) can be written in integral form as

$$\begin{aligned}\hat{v} &= \int_0^t \frac{e}{m_i}E_{\mathrm{T}}(\tau)\exp\{i\Omega_i(t-\tau)+i[\hat{\eta}(t)-\hat{\eta}(\tau)]\}\,\mathrm{d}\tau\\ &\quad+\int_0^t \frac{e}{m_i}\hat{E}_f(\tau)\exp\{i\Omega_i(t-\tau)+i[\hat{\eta}(t)-\hat{\eta}(\tau)]\}\,\mathrm{d}\tau+v_0\exp[i\Omega_i t+i\hat{\eta}(t)],\end{aligned} \tag{2}$$

where

$$\hat{\eta}(t) = \int_0^t \hat{\Omega}_f(\mathrm{s})\,\mathrm{d}s$$

and we have neglected terms proportional to $E_T B_T$ in the linear limit for the test wave.

To calculate the wave dispersion in a fluctuating magnetic field, we need to find the ion response to the electric field of the test wave, which is represented in the first term of (2). The second term does not affect the first as long as the statistical properties of the magnetic fluctuation $\hat{B}_f$ are unchanged. We shall discuss the self-consistency of this assumption in §5.

When $\eta(t) \ll 1$, we can approximate

$$\begin{aligned}\langle\exp\{i\Omega_i(t-\tau)+i[\hat{\eta}(t)-\hat{\eta}(\tau)]\}\rangle &\approx e^{i\Omega_i(t-\tau)}\{1+i\langle\hat{\eta}(t)\rangle-i\langle\hat{\eta}(\tau)\rangle\\ &\qquad-\tfrac{1}{2}\langle[\hat{\eta}(t)-\hat{\eta}(\tau)]^2\rangle\}\\ &= e^{i\Omega_i(t-\tau)}[1-\tfrac{1}{2}\langle\hat{\eta}(t)^2\rangle\\ &\qquad-\tfrac{1}{2}\langle\hat{\eta}(\tau)^2\rangle+\langle\hat{\eta}(t)\,\hat{\eta}(\tau)\rangle].\end{aligned} \tag{3}$$

We shall consider the validity of this approximation at the end of this section. With (3), we can write the ensemble-averaged drift velocity as

$$\langle \hat{v}_d \rangle = \frac{e}{m_i} \int_0^t E_{\mathrm{T}}(\tau)\, e^{i\Omega(t-\tau)} \left[1 - \tfrac{1}{2}\langle \hat{\eta}(\tau)^2 \rangle - \tfrac{1}{2}\langle \hat{\eta}(t)^2 \rangle + \langle \hat{\eta}(t)\, \hat{\eta}(\tau) \rangle\right] \mathrm{d}\tau, \tag{4}$$

where angular brackets denote an ensemble average. Now we give $\hat{\Omega}_f(t)$ as a time-stationary stochastic process with zero mean and autocorrelation $I(t)$, i.e.

$$\langle \hat{\Omega}_f(t) \rangle = 0, \quad \langle \hat{\Omega}_f(t_1)\, \hat{\Omega}_f(t_2) \rangle = I(t_1 - t_2). \tag{5}$$

Then the autocorrelation of η can be calculated as

$$\begin{aligned} \langle \hat{\eta}(t) \hat{\eta}(\tau) \rangle &= \int_0^t \mathrm{d}t' \int_0^\tau \mathrm{d}\tau' \langle \hat{\Omega}_f(t')\, \hat{\Omega}_f(\tau') \rangle = \int_0^t \mathrm{d}t' \int_0^\tau \mathrm{d}\tau' I(t' - \tau') \\ &= \int_0^t \mathrm{d}t' \int_0^\tau \mathrm{d}\tau' \int_{-\infty}^{\infty} \mathrm{d}\omega\, P(\omega)\, e^{i\omega(t'-\tau')} \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{P(\omega)}{\omega^2} (e^{i\omega t} - 1)(e^{-i\omega\tau} - 1), \end{aligned} \tag{6}$$

where $P(\omega)$ is the power spectrum of $\hat{\Omega}_f$ and we used the Wiener–Khinchine theorem $I(t) = \int d\omega P(\omega)\, e^{i\omega t}$.

Now we choose a circularly polarized test wave,

$$E_{\mathrm{T}}(t) = E_0\, e^{-i\omega_{\mathrm{T}} t}. \tag{7}$$

The ion drift caused by the above electric field is

$$\begin{aligned} \langle \hat{v}_d \rangle &= \frac{eE_0}{m_i} \int_0^t e^{-i\omega_{\mathrm{T}}\tau}\, e^{i\Omega_i(t-\tau)} \left[1 + \int_{-\infty}^{\infty} \frac{P(\omega)}{\omega^2} (e^{i\omega(t-\tau)} - i \sin \omega t + i \sin \omega\tau - 1)\, \mathrm{d}\omega\right] \mathrm{d}\tau \\ &= \frac{eE_0}{m_i} \int_0^t e^{-i\omega_{\mathrm{T}}\tau}\, e^{i\Omega_i(t-\tau)} \left\{1 + \int_{-\infty}^{\infty} \frac{P(\omega)}{\omega^2} [\cos \omega(t-\tau) -]\, \mathrm{d}\omega\right\} \mathrm{d}\tau \\ &= -\frac{ieE_0}{m_i} \left\{\frac{e^{-i\omega_{\mathrm{T}} t} - e^{i\Omega_i t}}{\Omega_i + \omega_{\mathrm{T}}} + \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{P(\omega)}{\omega^2} \frac{e^{-i\omega_{\mathrm{T}} t} - e^{i(\Omega_i - \omega)t}}{\Omega_i - \omega + \omega_{\mathrm{T}}} - \frac{P(\omega)}{\omega^2} \frac{e^{-i\omega_{\mathrm{T}} t} - e^{i\Omega_i t}}{\Omega_i + \omega_{\mathrm{T}}}\right] \mathrm{d}\omega\right\}. \end{aligned} \tag{8}$$

Here we have used the relation $P(\omega) = P(-\omega)$ to simplify the equation.

As an example suitable for analytical calculation, we assume a Lorentzian power spectrum in the form

$$P(\omega) = \frac{P_0 \gamma}{\pi(\omega^2 + \gamma^2)}. \tag{9}$$

We can then evaluate the integral, using the residue theorem, as

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{P_0 \gamma}{\omega^2 \pi(\omega^2 + \gamma^2)} \left(\frac{e^{i\omega_{\mathrm{T}} t} - e^{i(\Omega_i - \omega)t}}{\Omega_i - \omega + \omega_{\mathrm{T}}} - \frac{e^{i\omega_{\mathrm{T}} t} - e^{i\Omega_i t}}{\Omega_i + \omega_{\mathrm{T}}}\right) \mathrm{d}\omega \\ &\qquad = (\text{residue at } \omega = -i\gamma) + \tfrac{1}{2}\, (\text{residue at } \omega = 0). \end{aligned} \tag{10}$$

Since ω in (9) is a strictly real, we evaluate the integral as the Cauchy principal value; this is the reason for the factor $\frac{1}{2}$ multiplying the residue at $\omega = 0$. To obtain the above expression, we assumed that Im $(\omega_T) < 0$ from the physical reasoning that the test wave should not grow in this situation; this assumption is the cause of the time irreversibility of the solution. Using the above formula, we obtain the ion drift

$$\langle v_d \rangle = -\frac{eE_0}{m_i}\left\{\frac{i}{\Omega_i+\omega_{\mathrm{T}}}+\frac{iP_0}{2(\Omega_i+\omega_{\mathrm{T}})\,[(\Omega_i+\omega_{\mathrm{T}})^2+\gamma^2]}-\frac{P(\Omega_i+\omega_{\mathrm{T}})}{(\Omega_i+\omega_{\mathrm{T}})^2}\right\}e^{-i\omega_{\mathrm{T}}t}, \quad (11)$$

where we have dropped terms proportional to exp $(\Omega_i t)$, which will vanish on averaging over the gyro-phase.

The first term of (11) is the same as the ordinary cyclotron interaction without magnetic fluctuations. The second term represents the non-resonant effect of the magnetic field fluctuations. This term is purely imaginary, which implies that the drift velocity is $\frac{1}{2}\pi$ out of phase with the electric field. Therefore there is no net energy exchange due to this term. The third term is a real number, implying energy transfer from the test wave to ion motion. This term represents the parametric cyclotron resonance (PCR) discussed by Nakamura & Kennel (1993). We can see that this is the result of the resonance at $\omega = \Omega_i + \omega_T$ because it is equal to the residue at the pole of $\omega = \Omega_i + \omega_T$ (note that the residue at $\omega = -i\gamma$ is equal to the sum of residues at $\omega = \Omega_i + \omega_T$ and at $\omega = i\gamma$ for the terms proportional to exp $(-i\omega_T t)$).

The second term, reflecting the non-resonant effect of the magnetic fluctuations, depends on the overall shape of the power spectrum $P(\omega)$, while the third term, the resonant effect, depends only on the value of $P(\omega)$ near resonance. The ω integration in (8) will give the same expression for the resonant part as in (11) even for a differently shaped power spectrum, as long as it is a well-behaved analytic function near the resonance point.

When $\gamma \ll \Omega_i$ in (11), the resonant effect is negligible because the fluctuation components around the resonance are very small (note that the above calculation becomes invalid if γ is too small, because the condition (14) is not satisfied). The resonant and non-resonant effects are comparable when $\gamma \approx \Omega_i$. By using shifted a Lorentzian instead of (9), we can show that the resonant effect predominates if the spectrum is sharply peaked around the resonance point.

Before concluding this section, let us examine the validity of the approximation in (3). This equation is obtained by setting $\eta^m \approx 0$ for $m \gtrsim 3$; thus the condition for this approximation can be estimated as $\langle \eta^2 \rangle \ll 1$. Using (6), this condition can be expressed as

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{P(\omega)}{\omega^2} \ll 1. \quad (12)$$

For the Lorentzian spectrum in (9) we can write

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{P(\omega)}{\omega^2} = \frac{P_0}{2\gamma^2}. \quad (13)$$

On the other hand, $P_0 = \langle \hat{\Omega}_f^2 \rangle \, (= \langle \hat{B}_f^2 \rangle / B_0^2)$ from the Wiener–Khinchine theorem. Therefore the condition for the validity of the approximation in (3) is

$$\frac{\Omega_i^2}{\gamma^2}\frac{\langle \hat{B}_f^2 \rangle}{B_0^2} \ll 1 \quad (14)$$

for the Lorentzian power spectrum in (9).

3. Wave dispersion with magnetic fluctuations

In this section we calculate the dispersion relation of a monochromatic test wave whose frequency is much smaller than the electron Larmor and plasma frequencies. Since the fluctuation amplitude often decreases as the frequency increases in a turbulent plasma, we neglect the effects of the magnetic field fluctuations on electrons. Then we can describe electron perpendicular motion by the $\mathbf{E}\times\mathbf{B}$ drift only. When we neglect the temperature of the plasma, we obtain the total perpendicular current due to the test wave as

$$j_{\perp}(t)=\frac{E_0}{\mu_0 V_A^2}\left[\frac{i\Omega_i\omega_{\mathrm{T}}}{\Omega_i+\omega_{\mathrm{T}}}+iG(P(\,.\,);\Omega_i+\omega_{\mathrm{T}})-\frac{\Omega_i^2P(\Omega_i+\omega_{\mathrm{T}})}{(\Omega_i+\omega_{\mathrm{T}})^2}\right]e^{i\omega_{\mathrm{T}}t},\tag{15}$$

where we denote the term from the non-resonant effect by $G(P(\,.\,);\omega)$; this is a functional of the power spectrum $P(\omega)$. For example, $G(P(\,.\,);\omega)$ is defined as $G(P(\,.\,);\omega)\equiv\Omega_i^2P_0/[2\omega(\omega^2+\gamma^2)]$ for the Lorentzian distribution in (9). Since the power spectrum $P(\omega)$ is only defined for real values of ω, $P(\Omega_i+\omega_{\mathrm{T}})$ must be understood as an analytical continuation of it when ω_T is complex.

The currents in the x and y direction are given in matrix form by

$$\begin{pmatrix}j_x\\ j_y\end{pmatrix}=\frac{1}{\mu_0V_A^2}\begin{pmatrix}\dfrac{i\Omega_i^2\omega_{\mathrm{T}}}{\Omega_i^2-\omega_{\mathrm{T}}^2}+iG^--H^+ & \dfrac{\Omega_i\omega_{\mathrm{T}}^2}{\Omega_i^2-\omega_{\mathrm{T}}^2}+G^+-iH^-\\ -\dfrac{\Omega_i\omega_{\mathrm{T}}^2}{\Omega_i^2-\omega_{\mathrm{T}}^2}-G^++iH^- & \dfrac{i\Omega_i^2\omega_{\mathrm{T}}}{\Omega_i^2-\omega_{\mathrm{T}}^2}+iG^--H^+\end{pmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix},\tag{16}$$

where

$$G^{\pm}\equiv\tfrac{1}{2}[G(P(\,.\,);\Omega_i+\omega_{\mathrm{T}})\pm G(P(\,.\,);\Omega_i-\omega_{\mathrm{T}})],$$

$$H^{\pm}\equiv\frac{1}{2}\left[\frac{\Omega_i^2P(\Omega_i+\omega_{\mathrm{T}})}{(\Omega_i+\omega_{\mathrm{T}})^2}\pm\frac{\Omega_i^2P(\Omega_i-\omega_{\mathrm{T}})}{(\Omega_i-\omega_{\mathrm{T}})^2}\right].$$

The energy dissipation rate due to the electric field is

$$\overline{\mathbf{j}.\mathbf{E}}=\frac{H^+|\mathbf{E}|^2}{\mu_0V_A^2},\tag{17}$$

where the overbar represents an average over a time much longer than the wave period.

Let us seek the dispersion relation for a linearly polarized oblique propagating wave with its k vector in the (x,z) plane. Combining (15) with Maxwell's equations, we get the dispersion relation in the cold-plasma limit as

$$\begin{vmatrix}\dfrac{\Omega_i^2}{\Omega_i^2-\omega_{\mathrm{T}}^2}+\dfrac{G^--iH^+}{\omega_{\mathrm{T}}}-\dfrac{V_A^2k_z^2}{\omega_{\mathrm{T}}^2} & \dfrac{i\Omega_i\omega}{\Omega_i^2-\omega_{\mathrm{T}}^2}+\dfrac{iG^++H^-}{\omega_{\mathrm{T}}}\\ -\dfrac{i\Omega_i\omega_{\mathrm{T}}}{\Omega_i^2-\omega_{\mathrm{T}}^2}-\dfrac{iG^++H^-}{\omega_{\mathrm{T}}^2} & \dfrac{\Omega_i^2}{\Omega_i^2-\omega_{\mathrm{T}}^2}+\dfrac{G^--iH^+}{\omega_{\mathrm{T}}}-\dfrac{V_A^2k^2}{\omega_{\mathrm{T}}^2}\end{vmatrix}=0,\tag{18}$$

where we have ignored E_z, which is negligible in a cold plasma as long as the perpendicular scale length is much larger than the plasma skin depth.

4. Magnetic fluctuation effect on MHD waves

From the dispersion relation (18), we see that the magnetic fluctuation effect is more prominent for lower-frequency waves because the terms $(G^{\pm}+iH^{\pm})/\omega_T$ are larger at low frequencies. Therefore, even a small amplitude of magnetic

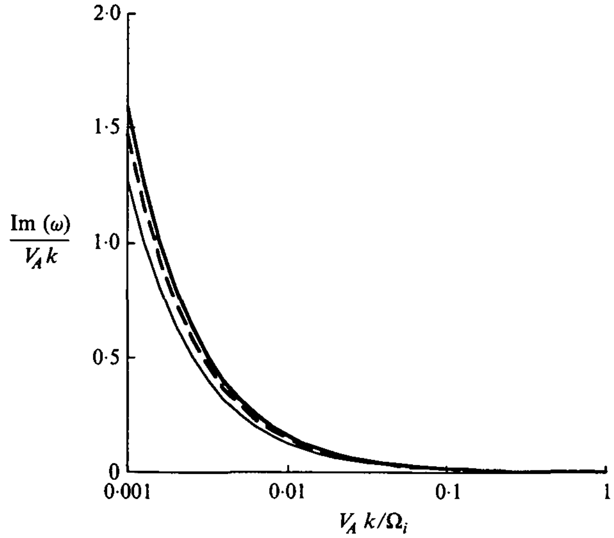

FIGURE 1. Damping rate of Alfvén waves due to magnetic fluctuations. The magnetic fluctuation is given as a Lorentzian power spectrum (9) with total amplitude $\langle \hat{B}_f^2/B_0^2 \rangle = 0{\cdot}01$ and width $\gamma = 0{\cdot}5$ (——), 1·0 (——) and 1·5 (---).

fluctuations can modify low-frequency MHD waves considerably. We examine two kinds of magnetic fluctuation effects in the following.

4.1. *MHD wave damping*

As we saw in §3, wave energy dissipates owing to resonance. As an example, let us calculate the dispersion relation for a circularly polarized parallel propagating MHD wave in a cold plasma from the current given in (15):

$$\omega_T^2 - V_A^2 k_z^2 + \omega_T[G(P(\,.\,);\Omega_i+\omega_T)+iP(\Omega_i+\omega_T)] = 0. \tag{19}$$

The imaginary part of the MHD wave frequency can be estimated as

$$\mathrm{Im}\,(\omega_T) \approx -P(\Omega_i+\omega_T). \tag{20}$$

The damping rate obtained here can be large even for small magnetic field fluctuations. For example, the Lorentzian power spectrum of (9) gives the variance of the magnetic field fluctuations as

$$\frac{\langle \hat{B}_f^2 \rangle}{B_0^2} = \frac{\langle \hat{\Omega}_f^2 \rangle}{\Omega_i^2} = \frac{1}{\Omega_i^2}\int_{-\infty}^{\infty} P(\omega)\,\mathrm{d}\omega = \frac{P_0}{\Omega_i^2}. \tag{21}$$

Then the damping rate for a broad spectrum with $\gamma \approx \Omega_i$ can be estimated as

$$\mathrm{Im}\,(\omega) \sim \frac{\langle \hat{B}_f^2 \rangle}{B_0^2}\frac{\Omega_i^2\gamma}{(\Omega_i^2+\gamma^2)} \sim \frac{\Omega_i}{2}\frac{\langle \hat{B}_f^2 \rangle}{B_0^2}. \tag{22}$$

Figure 1 is a plot of the damping rate against the wave frequency (normalized by ion gyrofrequency) with various γ.

4.2. *Mode coupling between shear mode and magnetosonic mode*

When the wave frequency cannot be neglected compared with the ion Larmor frequency, the Hall effect couples the two branches of MHD waves (fast mode and shear mode) in a cold plasma. For lower frequencies, the fast and shear wave are independent modes. However, magnetic field fluctuations can couple these two modes even if the wave frequency is much lower than the ion Larmor frequency. We examine this coupling using (18) in the low-frequency limit. If the power spectrum does not vary drastically within a small interval of frequency, we can approximate $G^+ \approx G(P(\,.\,);\Omega_i)$, $H^+ \approx P(\Omega_i)$ and $H^- \approx G^- \approx 0$. Then we get the dispersion relation for low-frequency waves from (18) as

$$\left(1-\frac{V_A^2 k_z^2}{\omega_\mathrm{T}^2}-\frac{iP(\Omega_i)}{\omega_\mathrm{T}}\right)\left(1-\frac{V_A^2 k^2}{\omega_\mathrm{T}^2}-\frac{iP(\Omega_i)}{\omega_\mathrm{T}}\right)=\left(\frac{\omega_\mathrm{T}}{\Omega_i}+\frac{G(P(\,.\,);\Omega_i)}{\omega_\mathrm{T}}\right)^2, \tag{23}$$

where we have neglected terms of higher order than $(\omega/\Omega_i)^2$. The term ω/Ω_i on the right-hand side of (23) represents the coupling due to the Hall effect, and the term next to it describes the coupling due to the magnetic fluctuations. We can estimate $G(P(\,.\,);\Omega_i)$ in the same way as in (22) for the Lorentzian spectrum of (9):

$$G(P(\,.\,);\Omega_i) \approx \frac{\langle \hat{B}_f^2 \rangle}{B_0^2}\frac{\Omega_i^3}{(\Omega_i^2+\gamma^2)}. \tag{24}$$

The coupling due to magnetic fluctuations becomes predominant when

$$\frac{\langle \hat{B}_f^2 \rangle}{B_0^2} > \frac{\omega^2}{\Omega_i^4}(\Omega_i^2+\gamma^2). \tag{25}$$

5. Waves in a finite-temperature plasma

Since we have neglected plasma temperature thus far, we have not been able to treat the slow magnetosonic mode. Here we take finite electron temperature into account to examine the effects of magnetic fluctuations on the fast and slow magnetosonic modes. It would also be interesting to examine the effects of finite ion temperature, such as those due to finite ion Larmor radius. However, this adds obscuring complexity that is beyond the scope of the present paper. Here we assume zero-temperature ions and finite-temperature electrons.

Again we prescribe a linearly polarized obliquely propagating MHD wave with its k vector in the (x,z) plane. Since the electron mobility in the ambient magnetic field direction is very large, the electron density perturbation is determined by the balance between the components of the pressure gradient and electric field in the parallel direction. From the equation of motion for the electron fluid, we get

$$\frac{T_e}{m_e n_0}\frac{\partial}{\partial z}\delta n_e = \frac{e}{m_e}E_z - \frac{\partial v_e}{\partial t} \approx \frac{e}{m_e}E_z, \tag{26}$$

where T_e is the electron temperature in energy units. If we assume that the anomalous drift due to magnetic fluctuations is much smaller than the $\mathbf{E}\times\mathbf{B}$ drift, we can write the ion perturbed density change in the form

$$\delta n_i = \frac{n_0}{\omega}\left(\frac{e}{m_i}\frac{k_z}{\omega}E_z+\frac{k_x}{B_0}E_y\right). \tag{27}$$

Assuming quasi-neutrality, we obtain the density perturbation from (26) and (27):

$$\delta n_i \approx \delta n_e \approx \frac{n_0 k_x}{\omega B_0}\left(1-\frac{V_S^2 k_z^2}{\omega_\mathrm{T}^2}\right)^{-1} E_y, \tag{28}$$

where V_S is the ion sound speed. The current in the y direction consists of two parts: the part from the ion drift given by (11) and the part from the diamagnetic drift of the electron fluid. We assume that the electron fluid has the same temperature in the parallel and perpendicular directions, and get

$$j_y = \frac{1}{\mu_0 V_A^2}\left\{\left[\frac{\Omega_i^2\,\omega_\mathrm{T}}{\Omega_i^2-\omega_\mathrm{T}^2} - G^- + iH^+ - \frac{V_S^2 k_x^2}{\omega_\mathrm{T}}\left(1-\frac{V_S^2 k_z^2}{\omega_\mathrm{T}^2}\right)^{-1}\right]E_y \right.$$
$$\left. + \left(\frac{\Omega_i\omega_\mathrm{T}^2}{\Omega_i^2-\omega_\mathrm{T}^2} + iG^+ - H^-\right)E_x\right\}. \quad (29)$$

There is no diamagnetic current in the x direction because $k_y = 0$; therefore the dispersion equation can be obtained as

$$\left(\frac{\Omega_i^2}{\Omega_i^2-\omega_\mathrm{T}^2} + \frac{G^- - iH^+}{\omega_\mathrm{T}} - \frac{V_A^2 k_z^2}{\omega_\mathrm{T}^2}\right)$$
$$\times\left[\frac{\Omega_i^2}{\Omega_i^2-\omega_\mathrm{T}^2} + \frac{G^- - iH^+}{\omega_\mathrm{T}} - \frac{V_S^2 k_x^2}{\omega_\mathrm{T}^2}\left(1-\frac{V_S^2 k_z^2}{\omega_\mathrm{T}^2}\right)^{-1} - \frac{V_A^2 k^2}{\omega_\mathrm{T}^2}\right]$$
$$-\left(\frac{i\Omega_i\,\omega_\mathrm{T}}{\Omega_i^2-\omega_\mathrm{T}^2} + \frac{iG^+ + H^-}{\omega_\mathrm{T}}\right)^2 = 0. \quad (30)$$

The first term in (30) is the product of the dispersion relations of the shear Alfvén mode and of the two magnetosonic modes, and the second term represents the coupling effect examined in §4. The coupling due to the Hall effect is negligible in the MHD frequency range. To simplify the problem, we also neglect the coupling due to the magnetic fluctuation, assuming $H^+ \gg G^+$ and $H^- \approx G^- \approx 0$. Then we can get the dispersion equation for magnetosonic modes as

$$\omega^4 - \omega_\mathrm{T}^2(V_S^2 + V_A^2)\,k^2 + V_A^2\,V_S^2\,k^2\,k_z^2 = (\omega_\mathrm{T}^2 - V_F^2\,k^2)(\omega_\mathrm{T}^2 - V_{Sl}^2\,k^2)$$
$$= iH^+\,\omega_\mathrm{T}(\omega_\mathrm{T}^2 - V_S^2\,k_z^2), \quad (31)$$

where V_F and V_{Sl} are the fast and slow magnetosonic speeds. The two magnetosonic modes are clearly coupled by the resonant interaction with the background field fluctuations.

6. Self-consistency

So far, we have simply prescribed a given power spectrum for the compressional magnetic fluctuations, and have not paid attention to the mechanism that maintains the fluctuations. Usually the fluctuation spectrum in plasmas is the result of complicated nonlinear effects, such as energy cascading. It is beyond the scope of the present paper to investigate such mechanisms, and we limit ourselves in this section to discussing the self-consistency of the existence of fluctuations.

In a turbulent plasma each fluctuation mode is not independent, and modes exchange energy. A mode that is slightly damped in linear theory can exist if energy is supplied from other modes. However, should a mode be strongly damped in linear theory, it is not realistic to assume that it plays a significant role in the plasma dynamics. Similarly, we have to examine the damping effect of magnetic field fluctuations on each wave component to check the self-consistency of our theory. We are especially interested in fluctuation modes whose frequencies are near the ion-cyclotron frequency, because these waves are important for the resonance. The non-resonant effect is a result of fluctuation modes over the entire frequency range, so it survives even when some components of a specific frequency suffer strong damping.

To obtain the dispersion relation (18), we did not limit the frequency of the test wave to the MHD range. Therefore we can consider the test wave to be one of the compressional waves that constitute the magnetic fluctuation spectrum. From (18), we see that there is no damping unless fluctuations exist around the frequency $\Omega_i \pm \omega$. Noting that the waves that contribute to the resonance have frequency $\omega \approx \pm \Omega_i$, we can say that the fluctuation components important for the resonance effect are stationary if the fluctuation components exist only around the ion-cyclotron frequency.

Even if fluctuations exist around the frequency $\Omega_i \pm \omega$, their effect on the waves around Ω_i is much smaller than on MHD waves. Suppose the terms estimated in (22) or (24) are of order the MHD wave frequency; then the magnetic fluctuation effect is quite strong on the MHD waves. On the other hand, its effect on a wave with $\omega \approx \Omega_i$ is very small, because the first term in each element of (18) is much larger; the damping rate due to magnetic fluctuations is comparable to the MHD wave frequency, which is much lower than the wave real frequency. Therefore the wave will exist long enough to modify the MHD wave even if energy is not supplied to the wave. As pointed out before, there should be some energy supply for each component of the fluctuation in a turbulent plasma, and it is plausible that this energy supply overcomes the small damping effect of the magnetic fluctuations.

We overlooked the magnetic field variation due to the MHD wave. In §5 we investigated MHD wave propagation in a fluctuating magnetic field, and the magnetic field variation due to the MHD wave itself was neglected in the linear approximation. An MHD wave with a finite compressional component of frequency ω_{MHD} damps fluctuation components at $\omega = \omega_{MHD} \pm \Omega_i$. These components are essential for damping the MHD wave itself, and so the MHD wave damping could be reduced. We cannot evaluate this effect without specifying the mechanism that sustains the magnetic fluctuations, but there is the possibility that the damping of the magnetosonic waves could be smaller than that of shear Alfvén waves.

There is another, more permissive way to view our calculations. Image one of the very large systems to which MHD theory is regularly applied, both in its analytical form and in the form of numerical simulations: the earth's magnetosphere. If we wish to obtain the properties of MHD waves whose wavelengths are the scale of this system, we should not overlook the fact that the waves probably include a number of independent microscopically turbulent domains and disjoint physical regimes within a wavelength. (Imagine a long-wavelength wave in the magnetosheath next to the impulsively reconnecting magnetopause.) Should we just ignore all the complexity that cannot be modelled? Certainly, the complexity will introduce a stochastic or chaotic aspect to the ensemble of single-particle orbits that underlies the computation of the MHD dispersion relation. Assuming that the magnitude of the magnetic field fluctuates is a way of saying that the Larmor phase in the ensemble of particle orbits has a certain indeterminate aspect in the real physical system. This indeterminacy need not be large before some of what we believe true about MHD waves is compromised.

7. Summary

We have studied the effects of high-frequency (of the order of the ion Larmor frequency) magnetic field fluctuations on MHD waves. Fluctuations in magnetic field strength cause a random variation of the Larmor frequency, and consequently an anomalous ion drift arises. The wave propagation will be modified when the current due to this anomalous drift is non-negligible in comparison with the ordinary perpendicular current of the wave. The ordinary current in low-frequency MHD waves comes from the polarization drift and pressure gradient drift, which is a very small portion of the total drift. Therefore the anomalous drift due to the magnetic fluctuation can have a significant effect on MHD waves even though the anomalous drift is small compared with the ordinary drift. To the best of our knowledge, the present paper is the first attempt to study this effect (which is why we have given very few references).

We have obtained the dispersion relation of waves in a fluctuating magnetic field and found the following:

(i) MHD waves suffer damping due to parametric cyclotron resonance if there are compressional fluctuation components near the ion-cyclotron frequency;

(ii) non-resonant magnetic fluctuations couple the shear and magnetosonic modes;

(iii) resonant fluctuations couple the fast and slow magnetosonic modes.

A simple estimate of these effects using a Lorentzian power spectrum shows that the magnetic fluctuation effects are significant even if their amplitude is much smaller than the ambient magnetic field.

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